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The visible set of a compactum in $\mathbb{R}^n$

Troy L. Goodsell [a], David G. Wright [b,*]

[a] *Department of Mathematics, Pittsburg State University, Pittsburg, KS 66762, USA*
[b] *Department of Mathematics, Brigham Young University, Provo, UT 84602, USA*

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Abstract

Let K be a compactum in $\mathbb{R}^n$. There is an arbitrarily small ambient isotopy of K so that if K' equals the image of K under the finishing homeomorphism then the points of K' which can be *seen from infinity* lie in a compact 0-dimensional subset of K'. © 2001 Elsevier Science B.V. All rights reserved.

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1. Definitions and notations

Let $\mathbb{R}^n$ denote Euclidean n-space with the usual metric.

Definition 1.1. If a is a point in $A \subset \mathbb{R}^n$ and k is a positive integer, a is said to be *k-visible with respect to A* if there exists a polygonal ray which lies in the union of k lines and meets A in exactly $\{a\}$. We say that the point a can be *seen from infinity* if it is 1-visible.

Definition 1.2. If $A \subset \mathbb{R}^n$ and k is a positive integer, then the *k-visible set* of A consists of all points in A that are k-visible. Notice, if A_k denotes the k-visible set of A, then $A_1 \subset A_2 \subset A_3 \subset \cdots$. The set $\bigcup A_k$ of points that are k-visible for some k is called the *visible set* of A. The set of points of A which do not lie in the visible set constitute the *invisible* set of A.

* Corresponding author.
E-mail addresses: tgoodsel@mail.pittstate.edu (T.L. Goodsell), wright@math.byu.edu (D.G. Wright).

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Definition 1.3. Two embeddings $f, g : K \to \mathbb{R}^n$ are said to be *ε-ambient isotopic* if there exists a homeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ such that h is isotopic to the identity by an isotopy that moves points less than ε and so that $g = h \circ f$. An *ε-homeomorphism* is a homeomorphism that moves points less than ε.

2. Main Theorem

The main result of this paper is the following theorem.

Theorem 2.1.
Let $f : K \to \mathbb{R}^n$ be an embedding of a compactum. Given $\varepsilon > 0$ and an integer $k > 0$ there exists an embedding $g : K \to \mathbb{R}^n$ such that f and g are ε-ambient isotopic and $g(K)$ has the property that the k-visible set lies in a compact 0-dimensional subset of $g(K)$.

Corollary 2.2. *Let $K \subset \mathbb{R}^n$ be a compactum. There is an arbitrarily small ambient isotopy of $\mathbb{R}^n$ taking K onto K' and a compact 0-dimensional $C \subset K'$ so that the projections of K' and C onto any hyperplane ($(n-1)$-dimensional subspace) are the same.*

Proof. Apply the theorem to K with $m = 1$ to get K'. Let C be a compact 0-dimensional set that contains the 1-visible set of K'. Then clearly the projections of K' and C onto any hyperplane ($(n-1)$-dimensional subspace) are the same. $\square$

See [2] for related results about projections of compacta in $\mathbb{R}^n$.

Meyerson [5] for $n = 3$ and Glaser [3] for $n > 3$ have constructed $(n-1)$-spheres in $\mathbb{R}^n$ so that each sphere S contains a tame Cantor set C with the property that S and C have the same projection in each direction. The compact 0-dimensional sets in our theorem are also tame as can be seen from the construction.

In Section 3 we give an interesting example of our technique with a simple closed curve in the plane. This example will be the basis for understanding the proof in higher dimensions. The main tool in our proof in higher dimensions is *strong general position* as introduced by Berkowitz and Roy [1] and covered in Section 4. We use this result to prove an even stronger theorem in Section 5 in the special case where K is a piecewise-linear n-manifold in $\mathbb{R}^n$. We then apply the theorem of Section 5 to a carefully constructed piecewise-linear n-manifold about an arbitrary compactum in Section 6.

3. The Salamander Curve

In this section we give a simple closed curve in the plane which we call the Salamander Curve which has the property that the 1-visible set lies in a Cantor set. Although this section is not used in the rest of the paper, the proof here illustrates nicely what is happening in higher dimensions.

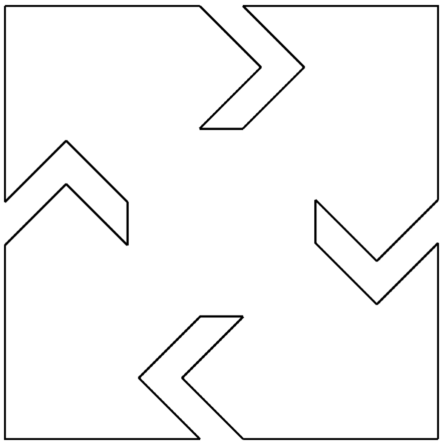

Fig. 1.

Theorem 3.1. *There is a simple closed curve J in the plane and a Cantor set $C \subset J$ so that the* 1*-visible set of J lies in C.*

Proof. Let K_0 be a square. We obtain K_1 from K_0 by pushing in a narrow tunnel at the midpoint of each edge. Each tunnel has a bend as illustrated in Fig. 1 so that the end of each tunnel is a closed interval that is not 1-visible with respect to K_1. We let W_1 denote the union of the intervals at the ends of the tunnels. Now at the midpoint of each edge of K_1, other than those that lie in W_1, push in a small narrow tunnel as shown in Fig. 2 to obtain K_2 so that the end of each tunnel is a straight line interval that is not 1-visible in K_2. We let W_2 denote the union of the intervals at the ends of these newer tunnels. Notice that $W_1 \cup W_2$ lies in K_2 and that no point of $W_1 \cup W_2$ is 1-visible with respect to K_2. Repeating this construction we get simple closed curves K_i and closed subsets W_i so that $W_1 \cup W_2 \cup \cdots \cup W_i \subset K_i$ and so that $W_1 \cup W_2 \cup \cdots \cup W_i$ is not 1-visible with respect to K_i. If $\varepsilon > 0$ is given and care is taken on the lengths of the tunnels, the curves K_i converge to a simple closed curve J so that there is a homeomorphism from K onto J that is ε-ambient isotopic to the identity. The set $\bigcup W_i$ is not 1-visible with respect to J. The 1-visible set of J lies in the closure of $J - \bigcup W_i$ which is a Cantor set. $\square$

The first two stages of the construction as illustrated in Fig. 2 suggests the name of *Salamander Curve* for this object.

The trick in higher dimensions is generalizing what we mean by the *bend* in the construction. This will be handled by the notion of Strong General Position which we will now introduce.

4. Strong general position

The key step in the construction of the Salamander Curve is the idea of the bend. To be more precise this can be thought of as taking three points and placing them in general

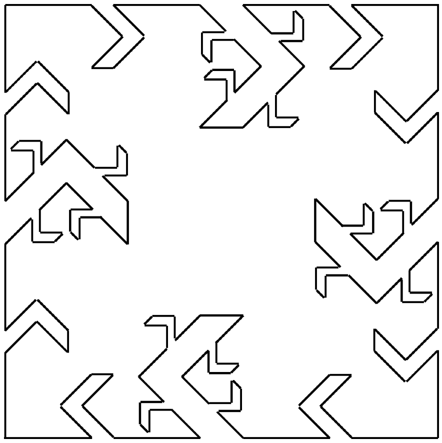

Fig. 2.

position so that no line intersects all three points. Notice that no line meets all three ε-balls about the three points for some small ε.

To do this in higher dimensions we need to work with codimension 2 polyhedra rather than just points. This is handled by the following theorem. Once we have the theorem guaranteeing that we cannot hit all the codimension 2 polyhedra with a straight line, we use Lemma 4.2 to get the further fact that any straight line cannot meet an epsilon neighborhood of each the polyhedra for some epsilon.

The first theorem is a special case of a theorem of Berkowitz and Roy [1]. A proof appears in [4].

Theorem 4.1.
Let P be a compact polyhedron in $\mathbb{R}^n$ of dimension less than or equal to $(n-2)$ so that P is the disjoint union of m subpolyhedra $P_1, P_2, \ldots, P_m$. Let $\varepsilon > 0$ be given. There is an ε piecewise-linear homeomorphism h of $\mathbb{R}^n$ fixed outside an ε neighborhood of P so that no line meets more than $2(n-1)$ of the $h(P_i)$.

Lemma 4.2. *If $A_1, A_2, \ldots, A_m$ is a collection of compact nonempty sets in $\mathbb{R}^n$ so that no line meets all of them, then there exists an $\varepsilon > 0$ so that the ε neighborhoods of the A_i also have the property that no line meets all of them.*

Proof. Let $A_1, A_2, \ldots, A_m$ be a collection of compact sets so that no line meets all of them. Then $A_1 \times A_2 \times \cdots \times A_{m-1}$ is a compact space. If $(a_1, a_2, \ldots, a_{m-1})$ is a point of $A_1 \times A_2 \times \cdots \times A_{m-1}$ let $\langle a_1, a_2, \ldots, a_{m-1}\rangle$ be the affine plane spanned by the points $a_1, a_2, \ldots, a_{m-1}$.

Notice that the dimension of $\langle a_1, a_2, \ldots, a_{m-1}\rangle$ is at least one; otherwise, by taking any point $a_m \in A_m$ we could get a line containing $a_1, a_2, \ldots, a_{m-1}, a_m$ which is a contradiction to the choice of the A_i.

Let A be the subset of $A_1 \times A_2 \times \cdots \times A_{m-1}$ consisting of all points $(a_1, a_2, \ldots, a_{m-1})$ so that the dimension of $\langle a_1, a_2, \ldots, a_{m-1}\rangle$ is exactly one. The set A can be seen to be a closed in $A_1 \times A_2 \times \cdots \times A_{m-1}$, because for any point $(a_1, a_2, \ldots, a_{m-1})$ in the complement of A, the dimension of $\langle a_1, a_2, \ldots, a_{m-1} \rangle$ is greater than or equal to two so that there are three points that are not collinear and they remain not collinear under a small shift.

Now define a map $f : A \to \mathbb{R}$ by $f(a_1, a_2, \ldots, a_{m-1})$ equals the distance from the line $\langle a_1, a_2, \ldots, a_{m-1} \rangle$ to the set A_m. Since A is a closed subset of a compact set, A itself is compact. Since f is continuous, it attains a minimum ε_m. Note that this minimum must be positive since the line $\langle a_1, a_2, \ldots, a_{m-1} \rangle$ meets each of the sets $A_1, A_2, \ldots, A_{m-1}$, and, therefore, cannot also meet A_m. Thus any line that meets each of $A_1, A_2, \ldots, A_{m-1}$ misses an ε_m neighborhood of A_m. Consider the closed $\varepsilon_m/2$ neighborhood A'_m of A_m. Notice that A'_m is compact and that the sets $A_1, A_2, \ldots, A_{m-1}, A'_m$ form a collection of compact sets so that no line meets all of them. So we repeat the argument on these sets using A_{m-1} in place of A_m. Thus we obtain ε_{m-1} and A'_{m-1} so that the sets $A_1, A_2, \ldots, A_{m-2}, A'_{m-1}, A'_m$ form a collection of compact sets so that no line meets all of them. By using the argument m times we get that the sets $A'_1, A'_2, \ldots, A'_{m-1}, A'_m$ form a collection of compact sets so that no line meets all of them. By setting ε equal to the smallest of the $\varepsilon_i/2$, we see that the ε neighborhoods of the A_i have the property that no line meets all of them. $\square$

By applying the above lemma repeatedly, we get the following corollary.

Corollary 4.3. *If $A_1, A_2, \ldots, A_m$ is a collection of compact nonempty sets in $\mathbb{R}^n$, then there exists an $\varepsilon > 0$ so that for each subset $\{i_1, i_2, \ldots, i_r\}$ of $\{1, 2, \ldots, m\}$ a line meets each of the ε neighborhoods of $A_{i_1}, A_{i_2}, \ldots, A_{i_r}$ if and only if there is a line that meets each of $A_{i_1}, A_{i_2}, \ldots, A_{i_r}$.*

5. Manifold case

We now set out to prove the Main Theorem in the special case where the compactum is a polyhedral n-manifold M in $\mathbb{R}^n$. We do this in two steps. We first prove a lemma that shows how to *hide* a neighborhood of an $(n-2)$-dimensional polyhedron in the boundary of M. The theorem which follows shows how to use this lemma repeatedly so as to get the desired Cantor sets.

Lemma 5.1. *Let M be a compact polyhedral n-manifold with boundary ∂M in $\mathbb{R}^n$ and P be an $(n-2)$-dimensional compact polyhedron in ∂M. Given $\varepsilon > 0$, and a positive integer k, then there exists a piecewise-linear homeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ so that the following conditions are satisfied*:

(1) *h is ε-isotopic to the identity,*
(2) *h is fixed on the complement of an ε neighborhood of P,*
(3) *there exists a neighborhood U of P in ∂M such that $h(U)$ does not meet the k-visible set of $h(M)$,*
(4) *there exists a $\delta > 0$ such that if $f : \mathbb{R}^n \to \mathbb{R}^n$ is a δ-homeomorphism then $f \circ h(U)$ does not meet the k-visible set of $f \circ h(M)$.*

Proof. We identify a polyhedral collar of ∂M in M with $\partial M \times [0,1]$ so that ∂M is identified with $\partial M \times \{0\}$. Let $m = 2k(n-1)+1$. This is the number needed to apply strong general position in our setting.

We assume without loss of generality that the collar $\partial M \times [0,1]$ is an $\varepsilon/2$ collar and that the m copies of P given by $P_i = P \times \{i/m\}$, $1 \leqslant i \leqslant m$, are in *strong general position* so that no line can meet more than $2(n-1)$ of the P_i. Notice that any polygonal path in $P \times [0,1]$ from a point of $P_0 = P \times \{0\}$ to a point of P_m must meet each P_i, $1 \leqslant i \leqslant m$. Furthermore, any such polygonal path must consist of at least $k+1$ intervals otherwise some line would meet more than $2(n-1)$ of the P_i.

Let ε' be the number given by Corollary 4.3 for the sets P_i, $1 \leqslant i \leqslant m$, so that a line can meet at most $2(n-1)$ of the ε'-neighborhoods of the P_i, denoted $N(P_i, \varepsilon')$. Choose a *small* regular neighborhood N of P in ∂M so that $N_i = N \times \{i/m\} \subset N(P_i, \varepsilon')$ for each i. Let N' be a smaller regular neighborhood of P in ∂M so that N' lies in the interior of N. We now let h be a piecewise-linear homeomorphism of $\mathbb{R}^n$ that is defined by pushing along the collar on ∂M. The homeomorphism h pushes N' all the way to $N' \times \{1\}$. This push is feathered out by a Urysohn function so that points in the complement of N are not pushed at all. If appropriate care is taken, this homeomorphism will satisfy conditions (1) and (2). The neighborhood U is N'. Any polygonal ray that begins at a point a of $h(U)$ and meets $h(M)$ exactly at $\{a\}$ contains a polygonal path that meets each N_i, $1 \leqslant i \leqslant m$, so the polygonal path must contain $k+1$ line segments and a is not k-visible with respect to $h(M)$. Choose $\delta > 0$ to be smaller than the distance from N_i to the complement of $N(P_i, \varepsilon')$ for each i. If $f : \mathbb{R}^n \to \mathbb{R}^n$ is a δ-homeomorphism, then any polygonal ray from a point b in $f \circ h(U)$ that meets $f \circ h(M)$ exactly at $\{b\}$ contains a polygonal path that meets each $N(P_i, \varepsilon')$, $1 \leqslant i \leqslant m$, so the polygonal path must contain $k+1$ line segments and b is not k-visible with respect to $f \circ h(M)$. □

Now we are ready for the proof of the following theorem:

Theorem 5.2. *Let M be a compact polyhedral n-manifold with boundary ∂M in $\mathbb{R}^n$. Then for each $\varepsilon > 0$ there exists a homeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ which is ε-isotopic to the identity and Cantor sets $C_1 \subset C_2 \subset \cdots$ each of which is tame in ∂M so that the k-visible set of $h(M)$ lies in $h(C_k)$.*

Proof. Let K be a triangulation of ∂M. We let P_i denote the $(n-2)$-skeleton of the ith barycentric subdivision of K and note that $P_i \subset P_{i+1}$ for each i. Apply the above lemma using $\varepsilon_1 < \varepsilon/2$ to the manifold M and polyhedron P_1 to get a homeomorphism h_1 and neighborhood U_1 of P_1 so that $h_1(U_1)$ does not meet the 1-visible set of $h_1(M)$. We let δ_1 be the delta given in condition (4) of the lemma. Assume without loss of generality that $\delta_1 < \varepsilon/2$.

Setting $\delta_0 = \varepsilon$, we may inductively assume for $1 \leqslant j \leqslant i$ we have:

(1) numbers $\delta_j > 0$ so that $\delta_j < \delta_{j-1}/2$,

(2) homeomorphisms $h_j : \mathbb{R}^n \to \mathbb{R}^n$ so that h_j is $(\delta_{j-1}/2)$-isotopic to the identity,

(3) neighborhoods U_j of P_j so that $h_j \circ \cdots \circ h_2 \circ h_1(U_j)$ does not meet the j-visible set of $h_j \circ \cdots \circ h_2 \circ h_1(M)$,

(4) if $f:\mathbb{R}^n \to \mathbb{R}^n$ is a δ_j homeomorphism, then $f \circ h_j \circ \cdots \circ h_2 \circ h_1(U_j)$ does not meet the j-visible set of $f \circ h_j \circ \cdots \circ h_2 \circ h_1(M)$.

Now apply the lemma with ε_{i+1} less than $\delta_i/2$ to the manifold $h_i \circ \cdots \circ h_2 \circ h_1(M)$ and polyhedron $h_i \circ \cdots \circ h_2 \circ h_1(P_{i+1})$ as the $(n-2)$-dimensional polyhedron in the boundary to get a homeomorphism h_{i+1} and neighborhood V_i of $h_i \circ \cdots \circ h_2 \circ h_1(P_{i+1})$ so that $h_{i+1} \circ h_i \circ \cdots \circ h_2 \circ h_1(V_i)$ does not meet the $(i+1)$-visible set of $h_{i+1} \circ h_i \circ \cdots \circ h_2 \circ h_1(M)$. Set U_{i+1} equal to the inverse of V_i under the homeomorphism $h_i \circ \cdots \circ h_2 \circ h_1$.

We let δ_{i+1} be the delta given in condition (4) of the lemma. It is important and we assume without loss of generality that $\delta_{i+1} < \delta_i/2$.

With further care in the choice of the ε_i, we may assume the sequence of homeomorphisms $(h_0, h_1 \circ h_0, h_2 \circ h_1 \circ h_0, \ldots)$ will converge to a homeomorphism h that is ε-isotopic to the identity.

By the choice of the δ_i, $h(U_j)$ also misses the j-visible set of $h(M)$. So the set $h(\bigcup_{i=j}^{\infty} U_i) = \bigcup_{i=j}^{\infty} h(U_i)$ misses the j-visible set of $h(M)$. Therefore setting $C_j = \bigcap_{i=j}^{\infty}(M - U_i)$, we have $h(C_j)$ contains the j-visible set of $h(M)$. But the closure of C_j misses P_i for each i. So the set C_j lies in a tame Cantor set of ∂M which we also denote by C_j. $\square$

It is interesting to note that for each i, $h(P_i)$ is invisible; i.e., it is not k-visible for any k. The visible set of $h(M)$ is 0-dimensional since it is contained in the countable union of closed 0-dimensional sets.

6. Proof of Main Theorem

Definition 6.1. A subset P of a topological space X is said to be *perfect* if it is closed and every point of P is a limit point of P.

Definition 6.2. Let I^n be the product of the interval $[-1, 1]$ n times.

Lemma 6.3. *Let $K \subset I^n \times [-1, 0] \subset I^n \times I$ be a closed set that is the union of a perfect set P and a countable set so that P meets the interior of $I^n \times \{0\}$. Then for any $\varepsilon > 0$ there is a homeomorphism $h: I^n \times I \to I^n \times I$, so that*

(1) $h(K) \subset I^n \times [-1, 0]$,

(2) $h(P) \cap$ *interior*$(I^n \times \{0\})$ *contains a tame Cantor set,*

(3) *h is ε-isotopic to the identity by an isotopy that keeps the boundary of $I^n \times I$ fixed.*

Proof. Let $P(0)$ be an element of $P \cap$ *interior*$(I^n \times \{0\})$. Let B be an $(n-1)$-ball in *interior*$(I^n \times \{0\})$ of radius less than 1 centered at $P(0)$.

If $P \cap$ *interior*(B) contains a point distinct from $P(0)$, call that point $P(1)$ and let h_1 be the identity homeomorphism on $I^n \times I$. Otherwise, choose $P(1)$ to be an element of *interior*$(B) - K$ distinct from $P(0)$ and a polygonal arc A from $P(1)$ to a point of P so that the length of A is less than $\varepsilon/2$ so that the arc meets K only at the end point in P. This can be accomplished because the point $P(0)$ is not an isolated point of P and $K - P$ is countable. Using the arc as a guide to push along, we construct a homeomorphism h_1 such that

(1) $h_1(P) \cap \mathit{interior}(I^n \times \{0\})$ contains $P(0)$ and $P(1)$,
(2) $h_1(K) \subset I^n \times [-1, 0]$,
(3) h_1 is $\varepsilon/2$-isotopic to the identity by an isotopy that keeps the boundary of $I^n \times I$ fixed.

We choose disjoint $(n-1)$-balls $B(0)$ and $B(1)$ in B centered at $P(0)$ and $P(1)$, respectively, with radius less than $\frac{1}{2}$.

We now set $P(0,0) = P(0)$ and $P(1,0) = P(1)$. We also choose distinct points $P(0,1) \in \mathit{interior}(B(0))$ and $P(1,1) \in \mathit{interior}(B(1))$. In a manner similar to the way h_1 was constructed, we now construct a homeomorphism h_2 so that

(1) $h_2(h_1(P)) \cap\ \mathit{interior}(I^n \times \{0\})$ contains $P(i,j)$, $0 \leqslant i, j \leqslant 1$,
(2) $h_2(h_1(K)) \subset I^n \times [-1, 0]$,
(3) h_2 is $\varepsilon/4$-isotopic to the identity by an isotopy that keeps the boundary of $I^n \times I$ fixed.

We continue by induction. Given $(n-1)$-balls $B(i_1, i_2, \ldots, i_{k-1})$, where $i_j \in \{0, 1\}$, containing points $P(i_1, i_2, \ldots, i_{k-1}, 0)$ and $P(i_1, i_2, \ldots, i_{k-1}, 1)$, choose disjoint $(n-1)$-balls $B(i_1, i_2, \ldots, i_{k-1}, 0), B(i_1, i_2, \ldots, i_{k-1}, 1) \subset B(i_1, i_2, \ldots, i_{k-1})$ of radius less than $1/(k+1)$ centered at the points $P(i_1, i_2, \ldots, i_k)$, respectively and a homeomorphism h_k of $I^n \times I$ to itself so that

(1) $h_k \circ \cdots \circ h_2 \circ h_1(P) \cap\ \mathit{interior}(I^n \times \{0\})$ contains $P(i_1, i_2, \ldots, i_k)$,
(2) $h_k \circ \cdots \circ h_2 \circ h_1(K) \subset I^n \times [-1, 0]$,
(3) h_k is $\varepsilon/2^{k+1}$-isotopic to the identity by an isotopy that keeps the boundary of $I^n \times I$ fixed.

If additional care is taken, then the limiting map $h = \lim_{k\to\infty} h_k \circ \cdots \circ h_2 \circ h_1$ is a homeomorphism that is ε-isotopic to the identity and the closure of $\{P(i_1, i_2, \ldots, i_k) \mid 0 \leqslant i_1, i_2, \ldots, i_k \leqslant 1 \text{ and } 1 \leqslant k < \infty\}$ is a tame Cantor set that lies in $h(P)$. Clearly, $h(K) \subset I^n \times [-1, 0]$. □

Lemma 6.4. *Given a compact uncountable set K in $\mathbb{R}^n$ and $\varepsilon > 0$, there is a piecewise-linear n-manifold M so that the boundary ∂M of M misses K, $K - M$ is countable, and for any tame Cantor set C in ∂M there exists a homeomorphism h of $\mathbb{R}^n$ so that*

(1) *$h(K \cap M) \subset M$,*
(2) *$C \subset h(K)$,*
(3) *h is ε-isotopic to the identity.*

Proof. The set K is the union of a perfect set P and a countable set. Let T be a fine triangulation of $\mathbb{R}^n$ so that each simplex in T has diameter less than $\eta = \varepsilon/4$. Let $\mathbb{R}$ be the polyhedron that is the union of all the closed simplexes of T that meet P. Let M be a regular neighborhood of $\mathbb{R}$ so that M lies in an η neighborhood of $\mathbb{R}$ and every point in ∂M can be joined to a point of P by a polygonal arc of length less than 2η. We may assume that ∂M misses K. Let C be a tame Cantor set in the boundary of M and let T_1 be a triangulation of the boundary of M so that each $(n-1)$-simplex, σ, in T_1 has diameter less than η and the boundary of each σ misses C. For each such σ, choose a polygonal arc A_σ of length less than 2η that lies in M, so that A_σ meets the boundary of M in one end point that lies in the interior of σ and meets K in the other end point which lies in P. We may assume that the polygonal arcs A_σ are disjoint. Using the polygonal arcs for guides, we construct a homeomorphism h_1 of $\mathbb{R}^n$ that is 2η-isotopic to the identity with $h_1(K \cap M) \subset M$ and so that $h_1(P)$ meets the interior of each $(n-1)$-simplex σ. By the previous lemma, there is a homeomorphism h_2 of $\mathbb{R}^n$ that is η-isotopic to the identity so that $h_2(h_1(K \cap M)) \subset M$ and so that the intersection of $h_2(h_1(P))$ with the interior of each $(n-1)$-simplex σ contains a tame Cantor set C_σ. For each σ that meets C, take an isotopy on σ that takes C_σ onto $C \cap \sigma$ fixing the boundary of σ. Using the collar on the boundary of M we extend the finishing homeomorphism of this isotopy to a homeomorphism h_3 of $\mathbb{R}^n$ so that that h_3 is η-isotopic to the identity. The homeomorphism h is obtained by setting $h = h_3 \circ h_2 \circ h_1$. □

We now give the proof of our Main Theorem.

Proof. We are given $f : K \to \mathbb{R}^n$ an embedding of a compactum, $\varepsilon > 0$, and an integer $k > 0$. If K is countable, there is nothing to prove. Otherwise, apply Lemma 6.4 with positive number $\varepsilon/2$ to the compactum $f(K)$ to get a manifold M. By Theorem 5.2 there is a tame Cantor set C in ∂M and a homeomorphism $h_2 : \mathbb{R}^n \to \mathbb{R}^n$ which is $\varepsilon/2$-isotopic to the identity and so that the k-visible set of $h_2(M)$ lies in $h_2(C)$. By Lemma 6.4 there is a homeomorphism $h_1 : \mathbb{R}^n \to \mathbb{R}^n$ so that

(1) $h_1(f(P)) \subset M$,
(2) $C \subset h_1(f(P))$,
(3) h_1 is $\varepsilon/2$-isotopic to the identity.

We now set $g = h_2 \circ h_1 \circ f$. Then f and g are ε-ambient isotopic. The k-visible set of $g(K)$ lies in $h_2(C) \cup h_2 \circ h_1(f(K) - M)$ which is a compact 0-dimensional subset of $g(K)$. □

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